ELECTROHYDRODYNAMIC FLOW IN A TWO-DIMENSIONAL CHANNEL WITH AN AXIALLY DISPOSED ELECTRODE-EMITTER

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A study is made of the flow of an incompressible nonviscous unipolar charged liquid in a two-dimensional channel $|x| < \infty$, $|y| \leq h$ with conducting walls and an axially disposed electrode-emitter (along $y=0$). The charged particles have an arbitrary constant mobility. The charge distribution on the emitter is approximated as a unit step function. The problem is solved by linearizing the equations with respect to the electrohydrodynamic interaction. The behavior of the electrical parameters is determined, and the deformed profiles of velocity and pressure downstream of the zone in which the electrostatic forces are rotational in character are calculated. These profiles can be determined without having to solve the linearized partial differential hydrodynamic equations in the entire region occupied by the flow, although the profiles then depend on the distribution of the electrical parameters along the entire length of the channel.

Electrohydrodynamic nonviscous flows in channels are usually investigated in the one-dimensional approximation (see [1, 2], for example). There are many devices, however, where the electric charge is introduced into the flow in such a manner that in a certain region of the flow the electrical forces are nonpotential, with the result that the flow acquires a spatial character (two-dimensional, in the simplest case). An example of this sort of flow is analyzed in [3], which deals with the motion of a liquid with zero charged-particle mobility in a two-dimensional channel with nonconducting walls. The perturbed motion of the liquid was determined for various laws describing the supply of charged particles by the electrode-emitter.

In addition to systems employing channels with nonconducting walls (when the emitter and the neutralizer are located in cross sections of the channel) there is also considerable interest in various other electrode arrangements: emitter grids and axially disposed needles, for example, to mention only two out of many. One of the possible schemes takes the form of a device with conducting walls and an axially disposed electrode-emitter. This scheme can simulate some of the processes which take place in corona devices.

In the present article we determine the electric fields and the asymptotic perturbed motion of an incompressible medium in a two-dimensional channel $|x| < \infty$, $y \leq |h|$ with conducting walls and an axially disposed (along $y=0$) electrode-emitter. As in [3] the liquid is assumed to be nonviscous, an assumption which is possible when the length of the working region of the channel is much greater than its transverse dimensions. The analysis is performed with the aid of perturbation theory (the electrohydrodynamic interaction is assumed small). The charged-particle mobility is arbitrary but constant.

1. We consider the two-dimensional motion of a nonviscous incompressible liquid in a channel $|x| < \infty$, $|y| \leq h$. Suppose that at $x = -\infty$ the flow conditions are uniform and the flow velocity is U (Fig. 1). We assume that the walls constitute electrodes which are at a potential below zero. An electrode emitting positive charges is situated in the middle of the channel (along the 0x axis). The electrode-emitter is sufficiently thin that it introduces no hydrodynamic perturbations into the flow.

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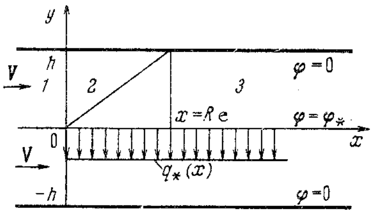

Fig. 1

As the processes in the upper and lower parts of the channel are symmetrical, we shall concern ourselves in future only with the upper region $0 \leq y \leq h$. The solution obtained can easily be modified to relate to the lower region.

The electrode-emitter is at a constant potential $\varphi_* > 0$, and under the action of the resulting electric field charge flows from the emitter into the passing liquid. A space-charge region is produced in the channel, the form of which depends on the flow velocity of the liquid and on the electric field (external and induced). In turn, the interaction of the liquid with the charged particles in this region causes the hydrodynamic parameters of the flow to vary. The resultant motion of the liquid is described by the equations of electrohydrodynamics

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+SqE_x, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+SqE_y \tag{1.1}$$

$$\frac{\partial}{\partial x}[q(uR_e+E_x)]+\frac{\partial}{\partial y}[q(vR_e+E_y)]=0$$

$$\Delta\varphi=-Nq, \qquad \mathbf{E}=-\operatorname{grad}\varphi \quad (\mathbf{j}=q\mathbf{V}R_e+q\mathbf{E}) \tag{1.2}$$

$$\left(\mathbf{V}=(u,v,0),\ \mathbf{E}=(E_x,E_y,0),\ R_e=\frac{Uh}{b\varphi_*},\quad S=\frac{q_*\varphi_*}{\rho U^2},\quad N=\frac{4\pi q_* h^2}{\varepsilon\varphi_*}\right)$$

Equations (1.1) and (1.2) are written in dimensionless form. The characteristic velocity, pressure, and dimension of the flow were taken to be U, ρU^2, and h (ρ is the density of the liquid). The characteristic values for φ (the potential of the electric field), **E**, q (electric-charge density), and j (electric-current density) are, respectively, φ_*, φ_*/h, q_*, and $bq_*\varphi_*/h$ (b is the mobility of the charged particles).

In order of magnitude the dimensionless criterion S equals the ratio of the electrostatic force to the force of inertia. The parameter N characterizes the magnitude of the induced electric field. The electric Reynolds number R_e determines, in order of magnitude, the ratio of convection current to conduction current.

We shall now formulate the boundary conditions subject to which (1.1) and (1.2) are to be solved. From the assumed uniformity of the flow at $x=-\infty$ and the fact that the liquid cannot pass through the electrodes, we have

$$u(-\infty,y)\equiv 1, \qquad v(x,1)=v(x,0)\equiv 0$$

$$\left(\int_0^1 u\,dy=1, \qquad x\in(-\infty,\infty)\right) \tag{1.3}$$

The boundary conditions for the electrical quantities are obvious:

$$\varphi(x,0)\equiv 1, \qquad \varphi(x,1)\equiv 0 \tag{1.4}$$

In addition, we must prescribe $q(x,0)=f(x)$, the charge-density distribution on the emitter. For definiteness we take $f(x)$ in the form

$$f(x)=\theta(x)=\begin{cases}0 & (x<0)\\ 1 & (x\geqslant 0)\end{cases} \tag{1.5}$$

The motion of the liquid is completely determined by equations (1.1) and (1.2) in conjunction with boundary conditions (1.3), (1.4), (1.5) and prescribed values of the dimensionless criteria S, N, R_e.

If the electrohydrodynamic interaction S is sufficiently small, the solution to the problem can be sought with the aid of perturbation theory. The electrical quantities are then found through the unperturbed distributions of the hydrodynamic parameters and satisfy the following system of equations:

$$\Delta\varphi=-Nq \qquad (\mathbf{E}=-\operatorname{grad}\varphi) \tag{1.6}$$

$$\frac{\partial q}{\partial x}\left(R_e-\frac{\partial\varphi}{\partial x}\right)-\frac{\partial q}{\partial y}\frac{\partial\varphi}{\partial y}=q\Delta\varphi$$

$$\varphi(x,0)\equiv 1, \quad \varphi(x,1)\equiv 0, \quad q(x,0)=\theta(x) \tag{1.7}$$

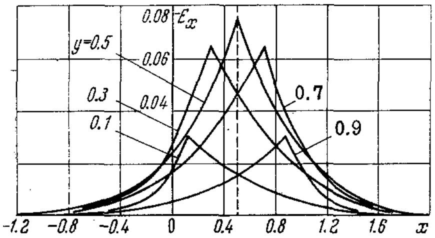

Fig. 2

Equations (1.6), (1.7) are obtained from (1.2) by putting in the latter $u \equiv 1$, $v \equiv 0$, $p = \text{const}$.

The equations describing the perturbations of the hydrodynamic parameters (the perturbations are indicated below by subscript "1") are obtained by linearizing (1.1) about the solution $u \equiv 1$, $v \equiv 0$, $p = \text{const}$:

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x} - Sq\frac{\partial \varphi}{\partial x}, \quad \frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y} - Sq\frac{\partial \varphi}{\partial y} \tag{1.8}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{1.9}$$

$$u_1(-\infty, y) \equiv 0, \quad v_1(x, 0) = v_1(x, 1) \equiv 0 \quad \left(\int_0^1 u_1\, dy \equiv 0\right) \tag{1.10}$$

The quantities q and φ appearing in (1.8) are the solution of the system (1.6), (1.7).

In future we shall be investigating the systems of equations (1.6), (1.7) and (1.8), (1.9).

2. We solve (1.6), (1.7) by the method of successive approximations. The zeroth approximation corresponds to no electric charges in the channel and, correspondingly, to zero induced fields. Consequently, we have for the zeroth approximation

$$q \equiv 0, \qquad \varphi = 1 - y \tag{2.1}$$

Let us now find the electric space charge in the first approximation. To this end we insert in Eq. (1.7) the zeroth approximation for the potential. The resulting equation

$$\frac{\partial q}{\partial x} R_e + \frac{\partial q}{\partial y} = 0 \tag{2.2}$$

was solved by the method of characteristics [4]. It has one family of characteristics which, starting from the points of the line $y = 0$, fill up all the space between the walls. Thus, prescribing q on the line $y = 0$ fixes the charge distribution everywhere.

For our chosen boundary condition, step function (1.5), the solution of (2.2) has the form

$$q = \theta(x - R_e y) = \begin{cases} 0 & (x < R_e y) \\ 1 & (x \geqslant R_e y) \end{cases} \tag{2.3}$$

The line $y = x/R_e$ is a boundary line for charge. The appearance of this line is connected with two factors: the removal of charges by the liquid flow and the motion of the charges in the external electric field. The inclination of this line to the 0x axis depends on the quantity R_e.

In this manner, three regions characterized by different charge distributions can be distinguished: region 1 ($x \leq 0$), where there is no charge; region 2 ($0 \leq x \leq R_e$), where space charge with a density equal to unity occupies only one half the region; and region 3 ($x \geq R_e$), where the charge density is everywhere equal to unity (Fig. 1).

Let us now calculate the potential in the first approximation. To this end we must substitute charge distribution (2.3) into Eq. (1.6) for the potential. We solve the resulting equation by introducing an auxiliary potential function $\psi = \varphi - 1 + y$, for which we obtain the equation

$$\Delta\psi = -N\,\theta(x - R_e y), \qquad \psi(x, 0) = \psi(x, 1) \equiv 0 \tag{2.4}$$

Equation (2.4) can be solved by the Fourier method, representing the sought function ψ in each of the regions 1, 2, 3 as a series of the form

$$\psi_i(x, y) = \sum_{k=1}^{\infty} a_{ik}(x) \sin(k\pi y) \qquad (i = 1, 2, 3) \tag{2.5}$$

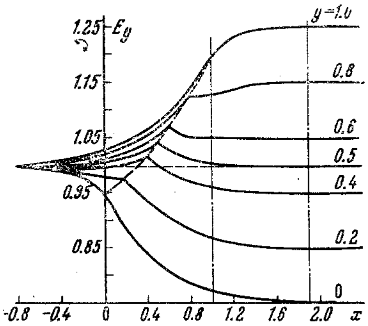

Fig. 3

On substituting (2.5) into (2.4), where the right side is also represented as a Fourier series, we obtain for the functions a_{ik} (x) ordinary differential equations whose boundary conditions follow from the requirements that the potential be bounded at infinity and that the normal and tangential components of the electric field be continuous at the boundaries between regions 1, 2, 3.

The final solution for φ has the form

$$\{x<0\}$$

$$\varphi_1 = 1 - y + \sum_{k=1}^{\infty} \sin(k\pi y) \sin^2\alpha \frac{N}{(k\pi)^3} e^{k\pi x}\{1 - \cos(k\pi) e^{-k\pi \operatorname{ctg}\alpha}\}$$

$$\{0 \leqslant x \leqslant \operatorname{ctg}\alpha\}$$

$$\varphi_2 = 1 - y + \sum_{k=1}^{\infty} \sin(k\pi y) \sin^2\alpha \frac{N}{(k\pi)^3} \left\{\frac{2}{\sin^2\alpha}[1 - \cos^2\alpha \cos(k\pi x \operatorname{tg}\alpha)] - \cos(k\pi) e^{k\pi(x - \operatorname{ctg}\alpha)} - e^{-k\pi x}\right\} \qquad (2.6)$$

$$\{x \geqslant \operatorname{ctg}\alpha\}$$

$$\varphi_3 = 1 - y + \sum_{k=1}^{\infty} \sin(k\pi y) \sin^2\alpha \frac{N}{(k\pi)^3}\left\{e^{k\pi(\operatorname{ctg}\alpha - x)} \cos(k\pi) - e^{-k\pi x} + \frac{2}{\sin^2\alpha}[1 - \cos(k\pi)]\right\}$$

$$(\operatorname{ctg}\alpha = R_e)$$

With the aid of this solution we can analyze the behavior of the electrical quantities within the entire channel in the presence of space charge.

Figure 2 shows the variation of E_x, the axial component of the electric field, along the length of the channel for various values of y = const. Figure 3 shows the variation of E_y, the transverse component of the electric field, along the length of the channel for various values of y = const. (We took N = 0.5, R_e = 1.) It can be seen that the maxima or kinks of the distributions all lie on the line y = x, i.e., all significant changes in the electrical quantities occur near the boundary of the space-charge region.

The charge distribution in the second approximation is found from Eq. (1.7), in which the potential φ and its derivatives are determined through formulas (2.6). The corresponding equation has the form

$$\frac{\partial q}{\partial x}\left(R_e - \frac{\partial\varphi}{\partial x}\right) - \frac{\partial q}{\partial y}\frac{\partial\varphi}{\partial y} = -qN\theta(x - yR_e), \qquad q(x, 0) = \theta(x) \qquad (2.7)$$

The characteristics of Eq. (2.7) are lines given by

$$\frac{dy}{dx} = -\frac{\partial\varphi / \partial y}{R_e - \partial\varphi / \partial x} \qquad (2.8)$$

The charge distribution on these characteristics is given by the expression

$$q = \exp\left\{-N\int_0^y \frac{\theta(x - yR_e)}{E_y} dy\right\} \qquad (2.9)$$

With the aid of (2.8) and (2.9) we can investigate how the space-charge region is deformed in comparison with the first approximation.

Let us estimate the precision of a successive-approximations calculation which terminates at formulas (2.8) and (2.9) for the electric charge. This we do by comparing the approximately computed parameters at x = ∞ with their precise values. (The latter can be determined by integration of ordinary differential equations, since at x = ∞ all electrical parameters depend on y alone.) The rate of convergence of the above method depends, of course, on the magnitude of the parameter N. Thus, the discrepancy between the approximate and the precise values of charge density at y = 1 for N = 0.5 proves to be less than 10%. If, however, N = 0.3, then the discrepancy does not exceed 5%.

3. We now determine the perturbations u_1, v_1, p_1 of the hydrodynamic parameters.

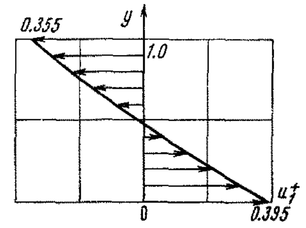

Fig. 4

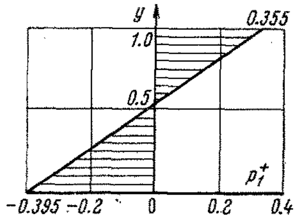

Fig. 5

There is no need to solve (1.8) and (1.9) within the entire region occupied by the flow in order to be able to calculate the perturbations obtaining far downstream from the edge of the space-charge zone. As shown in [3] such a calculation involves only ordinary differential equations. Let us find the appropriate differential equations in the present case. Remembering that at $x = \infty$ the parameters of the problem are independent of x, it follows from Eqs. (1.8) and (1.9) that

$$\frac{\partial v_1^+}{dy} = 0, \qquad \frac{dp_1^+}{dy} = Sq^+E_y^+ \tag{3.1}$$

(the index "+" denotes the value of the corresponding quantity at $x = \infty$).

Eliminating the pressure p_1 from (1.8), integrating the resulting expression with respect to x between the limits $(-\infty, x)$, and remembering that $q(-\infty, y) \equiv 0$, $u_1(-\infty, y) \equiv 0$, we obtain

$$\frac{du_1^+}{dy} = S\left[\frac{d}{dy}\int_{-\infty}^{\infty} qE_x\,dx - q^+E_y^+\right] \tag{3.2}$$

On the basis of solutions (2.6) and (2.9) we have

$$q^+ = \frac{2-N}{2Ny - N + 2}, \qquad E_y^+ = Ny + 1 - \frac{N}{2}, \qquad E_x^+ \equiv 0$$

Integrating (3.1) and (3.2) and applying boundary conditions (1.10), we obtain finally

$$v_1^+ \equiv 0, \quad u_1^+ = S\left(\frac{2-N}{4} - \frac{2-N}{2}y + \int_{-\infty}^{\infty} qE_x\,dx - \int_0^1\int_{-\infty}^{\infty} qE_x\,dx\,dy\right) \tag{3.3}$$

$$p_1^+ = S\left(\frac{2-N}{2}y - \frac{2-N}{4} + \int_0^1\int_{-\infty}^{\infty} qE_x\,dx\,dy\right) \tag{3.4}$$

These expressions determine the deformed velocity and pressure profiles at $x = -\infty$. It follows from (3.4) that the asymptotic flow is characterized by a nonuniform pressure distribution across the channel. Evidently, this is connected with the presence of transverse electric forces (there is no motion in the y direction). Note that the pressure at the wall $y = 1$ is greater than at the electrode-emitter $y = 0$, since the transverse component of the electric force is directed towards the wall away from the electrode-emitter. Integrating (3.4) over the cross section of the channel and remembering that $E_x \le 0$, we obtain

$$\int_0^1 p_1^+\,dy = S\int_0^1\int_{-\infty}^{\infty} qE_x\,dx\,dy < 0$$

The total pressure at the channel outlet is thus less than at the inlet. This reduction in pressure comes about because of the influence of axial electric forces, which are nonzero near the boundary of the space-charge region and which are directed against the flow.

Note that the distribution of the electrical quantities along the entire channel is required in order to calculate the asymptotic hydrodynamic parameters [see Eqs. (3.3) and (3.4)].

The relationships

$$u_1^+ = u_1^+(y), \; p^+{}_1 = p_1^+(y)$$

obtained from Eqs. (3.3) and (3.4) for $N = 0.5$, $R_e = 1$ are shown in Figs. 4, 5 respectively. The profile of $u^+(y)$ is characterized by large velocity values at the central electrode-emitter and by smaller values at the upper wall.

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